The Union-Closed Sets Conjecture

Bachelor Final Project

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1 Introduction

The *union-closed sets conjecture* (sometimes *Frankl's conjecture*, after Péter Frankl who posed the problem in 1979) is a beautiful open problem in the field of combinatorics. Suppose you are faced with a family of sets, for example

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{2,3\}\}.$$

Now consider the *union-closure* $\overline{\mathcal{F}}$ of $\mathcal{F}$, i.e. the family of all sets that can be formed as a *union* of any number of the sets in $\mathcal{F}$:

$$\overline{\mathcal{F}} = \{\emptyset, \{1\}, \{2\}, \{1,2\}, \{2,3\}, \{1,2,3\}\}.$$

The family $\overline{\mathcal{F}}$ is an example of a *union-closed* family; namely a family in which $A \in \mathcal{F}$ and $B \in \mathcal{F}$ together imply that $A \cup B \in \mathcal{F}$. The conjecture then claims the following:

> *In every finite union-closed family, some element belongs to at least half the member-sets of that family.*

Indeed, in our example both elements 1 and 2 appear in at least 3 out of 6 member-sets so that the conjecture is satisfied. Let us remark that $\mathcal{F} = \emptyset$ and $\mathcal{F} = \{\emptyset\}$ are obvious exceptions to the conjecture: they contain no elements. However, besides these two trivial families there are no known families for which the conjecture fails.

It is wonderful how the solution to a problem that can be stated in such elementary terms has eluded mathematicians for over thirty years. What makes the conjecture so difficult? In this paper, we give an overview of the results on the conjecture and look at related concepts.

1.1 Overview

Section 2 serves as an introduction of the concepts and definitions that are used in this paper. It is recommended that everyone should read it, in particular because some concepts are defined in a slightly different way than usual, in the light of the conjecture. Section 3 then covers different methods that have been employed in an attempt to prove the conjecture. Section 4 concerns statements of the conjecture in different mathematical fields. Finally, section 5 is a conclusion.

2 Basic concepts and definitions

In view of the union-closed sets conjecture, a number of definitions in this paper (can) mean something else than in other fields of mathematics. Words that have a 'special' meaning in this paper are, ordered alphabetically: *basis (set), common, dominate, dual, element, family, frequency, generate, ground set, member-set, neighbourhood, pair, rare, reduced, removable, separating, simple, trivial.*

Definition 1. A *family* $\mathcal{F}$ is a set in which the elements themselves are sets.

In this paper we shall refer to the sets of a family as *member-sets* and to the elements of those member-sets simply as *elements*.

Definition 2. A family $\mathcal{F}$ is *union-closed* if for any $A, B \in \mathcal{F}$ we have $A \cup B \in \mathcal{F}$.

Figure 1 shows an example of a union-closed family, where we have omitted commas and parentheses for readability.

$$\begin{array}{ccc} & 1234 & \\ 123 & 124 & 234 \\ 23 & 24 & 34 \\ & 3 \quad 4 & \\ & \emptyset & \end{array}$$

Figure 1: An example of a union-closed family.

Definition 3. We define the *ground set* of all elements that appear in some member-set of a family $\mathcal{F}$ by

$$X(\mathcal{F}) := \bigcup_{A \in \mathcal{F}} A.$$

If no confusion can arise, we may also simply refer to $X(\mathcal{F})$ as X. We also say that $\mathcal{F}$ is a family *on* X.

Definition 4. Given a family $\mathcal{F}$ and $x \in X$, we define the sub-family of all member-sets that contain x by

$$\mathcal{F}_x := \{A \in \mathcal{F} : x \in A\}.$$

Similarly, we define the sub-family of member-sets that do not contain x by $\mathcal{F}_{\overline{x}} := \mathcal{F} \setminus \mathcal{F}_x$. The *frequency* of x in $\mathcal{F}$ is $|\mathcal{F}_x|$. We extend definition 4 in a straightforward manner to cover multiple elements, for instance $\mathcal{F}_{\overline{x}y} = \mathcal{F}_{\overline{x}} \cap \mathcal{F}_y$.

Proposition 1. *Let $\mathcal{F}$ be a union-closed family and $x \in X$. Then both $\mathcal{F}_x$ and $\mathcal{F}_{\overline{x}}$ are union-closed.*

Proof. Let $A, B \in \mathcal{F}_x$. Then $A, B \in \mathcal{F}$ and therefore $A \cup B \in \mathcal{F}$. Since $x \in A, B$ we also have $x \in A \cup B$, so $A \cup B \in \mathcal{F}_x$. By definition 2, $\mathcal{F}_x$ is union-closed. The proof for $\mathcal{F}_{\overline{x}}$ is analogous. $\square$

Definition 5. An element $x \in X$ is *common* in a family $\mathcal{F}$ if $|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|$.

Definition 6. An element $x \in X$ is *rare* in a family $\mathcal{F}$ if $|\mathcal{F}_x| \leq \frac{1}{2}|\mathcal{F}|$.

We say a union-closed family $\mathcal{F}$ is *trivial* if $\mathcal{F} = \emptyset$ or $\mathcal{F} = \{\emptyset\}$. We can now state the central topic of this paper:

Conjecture 1. Union-closed sets conjecture. *Any finite non-trivial union-closed family contains a common element.*

In view of this conjecture, all families considered in this paper are assumed to be finite. We have to exclude the trivial families from the conjecture because they do not contain any elements. In the remainder of the paper, we will refer to conjecture 1 as the *main conjecture*.

At first, the requirement that an element be contained in at least 1/2 of all member-sets may seem rather arbitrary. Why not use a different lower bound? Primarily because this is the highest we can go: power sets are an example for which the bound is tight. However, another interesting thing to see here is that $|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|$ if and only if $|\mathcal{F}_x| \geq |\mathcal{F}_{\overline{x}}|$. In particular, an injection then exists that maps $\mathcal{F}_{\overline{x}}$ into $\mathcal{F}_x$. In section 3.1 we cover these injections.

Definition 7. A non-empty member-set A of a family $\mathcal{F}$ is a *basis set* if there are no member-sets $B, C \in \mathcal{F} \setminus \{A\}$ such that $B \cup C = A$.

In other words, the basis sets of a union-closed family $\mathcal{F}$ are precisely the sets that cannot be formed as a union of proper subsets in $\mathcal{F}$.

Definition 8. Given a family $\mathcal{F}$, the *union-closure* of $\mathcal{F}$ is the union-closed family $\overline{\mathcal{F}}$ defined by

$$\overline{\mathcal{F}} = \left\{ \bigcup_{A \in \mathcal{F}'} A : \mathcal{F}' \subseteq \mathcal{F} \right\}.$$

Note that $\emptyset \in \overline{\mathcal{F}}$. Intuitively speaking, the elements in $\mathcal{F}$ serve as the building blocks for the union-closure $\overline{\mathcal{F}}$. We also say that $\overline{\mathcal{F}}$ is *generated* by $\mathcal{F}$. This gives rise to the idea of a generating family that is minimal under inclusion:

Definition 9. Given a union-closed family $\mathcal{F}$, we define the *basis* of $\mathcal{F}$ by

$$\mathcal{B}(\mathcal{F}) \coloneqq \{A \in \mathcal{F} : A \text{ is a basis set}\}.$$

If no confusion can arise, we may also simply refer to $\mathcal{B}(\mathcal{F})$ as $\mathcal{B}$. Bruhn and Schaudt [1] motivate the notion of a basis and basis sets with the following theorem:

Theorem 1. *Let $\mathcal{F}$ be a union-closed family with basis $\mathcal{B}$. Then the union-closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ is equal to $\mathcal{F} \cup \{\emptyset\}$. Moreover, $\mathcal{B}$ is minimal under inclusion with respect to this property.*

Proof. Since $\mathcal{B} \subseteq \mathcal{F}$ and any member-set in $\overline{\mathcal{B}}$ is a union of (possibly zero) member-sets in $\mathcal{B}$, we trivially have $\overline{\mathcal{B}} \subseteq \mathcal{F} \cup \{\emptyset\}$. Now let $A \in \mathcal{F} \cup \{\emptyset\}$. If A is a basis set or the empty set, then clearly $A \in \overline{\mathcal{B}}$. So assume A is non-empty and not a basis set. Then A is the union of two other member-sets that are strictly smaller than A. By repeatedly applying this argument we find that A is the union of a number of basis sets sets. It follows that $\overline{\mathcal{B}} = \mathcal{F} \cup \{\emptyset\}$. The minimality of $\mathcal{B}$ follows directly from the definition. After all, none of the basis sets can be formed as a union of other member-sets, so the closure of any proper subset of $\mathcal{B}$ does not contain all the basis sets. □

As we shall see shortly, we may always assume the empty set to be a member-set of a family without loss of generality w.r.t the main conjecture. Then theorem 1 implies that any union-closed family is exactly the union-closure of its basis.

Corollary 1. *Let $\mathcal{F}$ be a union-closed family and $x \in X$. Then there exists an $A \in \mathcal{B}$ such that $x \in A$.*

Proof. Suppose, in order to reach a contradiction, that there exists $x \in X(\mathcal{F})$ such that $x \notin A$ for all $A \in \mathcal{B}$. Then none of the member-sets in the union-closure $\overline{\mathcal{B}}$ of $\mathcal{B}$ contain x and so $x \notin X(\overline{\mathcal{B}})$ by definition 3. However, this is exactly equal to $X(\mathcal{F})$ and so we have reached a contradiction. The result then follows. □

As an example, consider again the union-closed family of figure 1. The basis of that family is

$$\mathcal{B} = \{\{1,2,3\},\{1,2,4\},\{2,3\},\{2,4\},\{3\},\{4\}\}.$$

Bruhn and Schaudt [1] also note a very useful property of basis sets:

Proposition 2. *Let $\mathcal{F}$ be a union-closed family and $A \in \mathcal{F}$ non-empty. Then the family $\mathcal{F} \setminus \{A\}$ is union-closed if and only if A is a basis set.*

Proof. Let A be a basis set. If $\mathcal{F} \setminus \{A\}$ is empty, it is trivially union-closed. Therefore, assume otherwise and let $B, C \in \mathcal{F} \setminus \{A\}$. Then $B \cup C \in \mathcal{F}$ by the union-closedness of $\mathcal{F}$. Moreover, $B \cup C \neq A$ as A is a basis set. Hence $B \cup C \in \mathcal{F} \setminus \{A\}$ and $\mathcal{F} \setminus \{A\}$ is union-closed. Now let A be non-basis set. Then there exist $B, C \in \mathcal{F} \setminus \{A\}$ such that $B \cup C = A$. But then $B \cup C \notin \mathcal{F} \setminus \{A\}$ and $\mathcal{F} \setminus \{A\}$ is not union-closed. □

The following observation provides a lower bound for the number of basis sets of a union-closed family.

Proposition 3. *Let $\mathcal{F}$ be a union-closed family. Then its basis $\mathcal{B}$ has at least size $|\mathcal{B}| \geq \log_2 |\mathcal{F}|$.*

Proof. From definition 9, the union-closure $\mathcal{F}$ of a basis with size $|\mathcal{B}|$ contains at most $2^{|\mathcal{B}|}$ distinct member-sets. Therefore $2^{|\mathcal{B}|} \geq |\mathcal{F}|$. The result follows by taking logarithms on both sides. □

Definition 10. Given a family $\mathcal{F}$ and $x \in X$, we define by

$$\mathcal{F} - x \coloneqq \{A \setminus \{x\} : A \in \mathcal{F}\}$$

the family obtained by deleting x from each of its member-sets.

Proposition 4. *Let $\mathcal{F}$ be a union-closed family and $x \in X$. Then $\mathcal{F} - x$ is union closed.*

Proof. Let $A', B' \in \mathcal{F} - x$. Then there exist $A, B \in \mathcal{F}$ such that $A \setminus \{x\} = A'$ and $B \setminus \{x\} = B'$. By union-closedness of $\mathcal{F}$ we have $A \cup B \in \mathcal{F}$. Then $A' \cup B' = (A \cup B) \setminus \{x\} \in \mathcal{F} - x$. By definition 2, $\mathcal{F} - x$ is union-closed. □

Some union-closed families have properties that can be used to reduce the family to a 'simpler' family that has similar characteristics:

Proposition 5. *Let $\mathcal{F}$ be a non-trivial union-closed family and $\emptyset \notin \mathcal{F}$. Then $\mathcal{F}$ satisfies the main conjecture if $\mathcal{F} \cup \{\emptyset\}$ does.*

Proof. Let $x \in X$. Then the frequency of x in $\mathcal{F} \cup \{\emptyset\}$ is equal to the frequency of x in $\mathcal{F}$. Moreover, the family $\mathcal{F} \cup \{\emptyset\}$ contains one more member-set than $\mathcal{F}$. Then x is common in $\mathcal{F}$ if x is common in $\mathcal{F} \cup \{\emptyset\}$. □

From proposition 5 it is seen that we may assume $\emptyset \in \mathcal{F}$ for all families $\mathcal{F}$ without loss of generality w.r.t. the main conjecture.

Definition 11. Two distinct elements $x, y \in X$ form a *pair* in a family $\mathcal{F}$ if $\mathcal{F}_x = \mathcal{F}_y$. A family $\mathcal{F}$ is *separating* if it contains no pairs.

As many authors remark, we may assume all families to be separating without loss of generality w.r.t. the main conjecture. In fact, internet user kevkev1695 [2] recently observed that we may assume an even stronger property:

Definition 12. An element $x \in X$ is *removable* in a family $\mathcal{F}$ if $|\mathcal{F} - x| = |\mathcal{F}|$. A family $\mathcal{F}$ is *reduced* if it contains no removable elements.

Whether an element is removable from a union-closed family can be determined from the following sub-family:

Definition 13. Given a family $\mathcal{F}$ and $x \in X$, we define by

$$\mathcal{S}_x(\mathcal{F}) \coloneqq \{A \in \mathcal{F}_x : A \setminus \{x\} \in \mathcal{F}\}$$

the sub-family of member-sets so that deleting x from such a set yields again a member-set.

If no confusion can arise, we may also simply refer to $\mathcal{S}_x(\mathcal{F})$ as $\mathcal{S}_x$.

Proposition 6. *Let $\mathcal{F}$ be a union-closed family and $x \in X$. Then x is removable if and only if $\mathcal{S}_x$ is empty.*

Proof. The mapping defined by

$$\begin{aligned}\varphi : \mathcal{F} &\longrightarrow \mathcal{F} - x\\ A &\longmapsto A \setminus \{x\}\end{aligned}$$

is clearly surjective. Moreover, it is injective if and only if S_x is empty. □

In other words, removable elements are precisely the elements that we can delete from a family without getting double member-sets. It is now easy to prove that being reduced implies being separating.

Proposition 7. *Let $\mathcal{F}$ be a reduced union-closed family. Then $\mathcal{F}$ is separating.*

Proof. Let $\mathcal{F}$ be a non-separating family and let $x, y \in X$ be a pair. Then for all $A \in \mathcal{F}_x$ we have that $y \in A \setminus \{x\}$, so that $A \setminus \{x\} \notin \mathcal{F}$ by definition 11. But then x is removable by proposition 6 and $\mathcal{F}$ is not reduced. The result follows by contraposition. □

In particular, the family $\mathcal{F}$ defined by

$$\mathcal{F} := \{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$$

is an example of a separating family that is not reduced, as 2 is a removable element. Thus, being reduced is indeed stronger than being separating.

Theorem 2. *Let $\mathcal{F}$ be a union-closed family and $x \in X$ a removable element. Then $\mathcal{F}$ satisfies the main conjecture if $\mathcal{F} - x$ does.*

Proof. Let $y \in X$ such that $y \neq x$. Then we have $|(\mathcal{F} - x)_y| = |\mathcal{F}_y|$ by definition 12. Together with $|\mathcal{F} - x| = |\mathcal{F}|$, this implies that y is common in $\mathcal{F} - x$ if and only if y is common in $\mathcal{F}$. □

Hence, from repeated application of theorem 2 it is seen that we may assume all families to be reduced (and therefore separating) without loss of generality w.r.t. the main conjecture.

Finally, we remark that the sub-families $\mathcal{S}_x$ of a union-closed family are again union-closed:

Proposition 8. *Let $\mathcal{F}$ be a union-closed family and $x \in X$. Then $\mathcal{S}_x$ is union-closed.*

Proof. Let $A, B \in \mathcal{S}_x$. Then $A, B \in \mathcal{F}_x$ and therefore $A \cup B \in \mathcal{F}_x$ by the union-closedness of $\mathcal{F}_x$. Moreover, since $A \setminus \{x\} \in \mathcal{F}$ and $B \setminus \{x\} \in \mathcal{F}$ we have $(A \cup B) \setminus \{x\} \in \mathcal{F}$. It follows that $A \cup B \in \mathcal{S}_x$. By definition 2, $\mathcal{S}_x$ is union-closed. □

3 Methods of proof

One may attempt to prove (or disprove) the union-closed sets conjecture in a great variety of ways. In this chapter we cover three approaches to the conjecture and their insightful results: injections, averaging, and counter-examples.

3.1 Injections

Let $\mathcal{F}$ be a non-trivial union-closed family. For any element $x \in X$, the family $\mathcal{F}$ is then partitioned into $\mathcal{F}_x$ and $\mathcal{F}_{\overline{x}}$. Hence x is common in $\mathcal{F}$ if and only if $|\mathcal{F}_{\overline{x}}| \leq |\mathcal{F}_x|$. Equivalently, there exists some *injection* that maps $\mathcal{F}_{\overline{x}}$ into $\mathcal{F}_x$.

For some families, such an injection is easy to construct. One result that follows from this method proves the conjecture for union-closed families that contain a singleton, i.e. a set that contains only one element:

Theorem 3. *Let $\mathcal{F}$ be a union-closed family that contains a member-set $\{x\}$. Then x is common in $\mathcal{F}$.*

Proof. The result follows from the injection defined by

$$\begin{aligned} \varphi : \mathcal{F}_{\overline{x}} &\longrightarrow \mathcal{F}_x \\ A &\longmapsto A \cup \{x\} \end{aligned}$$

□

Similarly, one obtains a result for union-closed families containing a member-set of two elements:

Theorem 4. *Let $\mathcal{F}$ be a union-closed family that contains a member-set $\{x, y\}$. Then at least one of x, y is common in $\mathcal{F}$.*

Proof. The injection defined by

$$\begin{aligned} \varphi : \mathcal{F}_{\overline{xy}} &\longrightarrow \mathcal{F}_{xy} \\ A &\longmapsto A \cup \{x, y\} \end{aligned}$$

implies $|\mathcal{F}_{xy}| \geq |\mathcal{F}_{\overline{xy}}|$. Additionally, by symmetry, assume $|\mathcal{F}_{x\overline{y}}| \geq |\mathcal{F}_{\overline{x}y}|$. Then

$$|\mathcal{F}_x| = |\mathcal{F}_{x\overline{y}}| + |\mathcal{F}_{xy}| \geq |\mathcal{F}_{\overline{x}y}| + |\mathcal{F}_{\overline{xy}}| = |\mathcal{F}_{\overline{x}}|.$$

Hence x is common in $\mathcal{F}$. □

Sadly we do not have a similar result for families containing a member-set of three elements. In fact, Poonen [3] constructs a union-closed family that contains a unique smallest non-empty member-set of three elements, none of which is common. See figure 2 (note that this is not a counter-example to the main conjecture itself, but merely to a generalization of theorems 3 and 4).

The greatest obstacle to the method of injections is that we often don't know where to expect the common elements. As figure 2 illustrates, it is not guaranteed that the smallest member-set of a family contains a common element. This complicates the process of finding a suitable injection.

123456789

12345678 12345679 12345689 12345789 12346789
12356789 12456789 13456789 23456789

1456789 2456789 3456789

146789 156789 245689 245789 345678 345679 456789

45678 45679 45689 45789 46789 56789

123

∅

Figure 2: Each element of {1,2,3} is contained in only 13 out of 28 member-sets. For details on its construction, see Poonen's article [3].

3.2 Averaging

Let $\mathcal{F}$ be a non-trivial union-closed family. If the *average element frequency* is at least half the size of $\mathcal{F}$, then we clearly must have a common element. In other words, if

$$\frac{1}{|X|}\sum_{x\in X}|\mathcal{F}_x| \geq \frac{1}{2}|\mathcal{F}|, \tag{1}$$

then $\mathcal{F}$ satisfies the main conjecture.

Of course, the individual frequencies $|\mathcal{F}_x|$ are generally not known. After all, the conjecture would otherwise not be that difficult. However, Bruhn and Schaudt [1] remark that a double-counting argument provides a very useful identity. Indeed, counting all pairs (x, A) with $x \in A \in \mathcal{F}$ in two different ways yields:

$$\sum_{x\in X}|\mathcal{F}_x| = \sum_{A\in\mathcal{F}}|A|. \tag{2}$$

Substituting (2) into (1) then yields the equivalent condition

$$\frac{1}{|\mathcal{F}|}\sum_{A\in\mathcal{F}}|A| \geq \frac{1}{2}|X|. \tag{3}$$

Thus, if the *average member-set size* is at least half the size of the ground set, then $\mathcal{F}$ satisfies the main conjecture.

Bruhn and Schaudt proceed to show that condition (3) has lead to strong results on the conjecture. We cover some of these results below to illustrate the strength of the averaging method. For a more comprehensive overview, see their article [1].

The following theorem is attributed to Nishimura and Takahashi, although I have been unable to get a hold of their article. It says that a union-closed family always satisfies the conjecture, if it is large enough in comparison with the number of elements in its ground set. Stronger bounds have been determined, but we mention this result because the proof as presented by Bruhn and Schaudt is an excellent example of the averaging method:

Theorem 5. [1] *Let $\mathcal{F}$ be a union-closed family on m elements. If $\mathcal{F}$ contains more than $2^m - \frac{1}{2}\sqrt{2^m}$ member-sets, it satisfies conjecture 1.*

Proof. Suppose, in order to reach a contradiction, that there exists a set $S \subseteq X$ such that $S \notin \mathcal{F}$ but $|S| \geq \frac{m}{2}$. For any $R \subseteq S$ with $R \in \mathcal{F}$, we have $S \setminus R \notin \mathcal{F}$ since $(S \setminus R) \cup R = S \notin \mathcal{F}$. Thus, at least half of the subsets of S are missing in $\mathcal{F}$. But this implies $|\mathcal{F}| \leq 2^m - \frac{1}{2}2^{\frac{m}{2}}$, a contradiction. So every set $S \subseteq X$ containing at least $\frac{m}{2}$ elements must be a member-set of $\mathcal{F}$. Then the average member-set size is at least $\frac{m}{2}$ and condition (3) is satisfied, from which the result follows. □

As we mentioned, this bound may be improved. The strongest bound on the size of $\mathcal{F}$ given a fixed number of elements is due to Balla et al. [4]:

Theorem 6. [4] *Let $\mathcal{F}$ be a union-closed family on m elements. If $\mathcal{F}$ contains at least $\left\lceil \frac{2}{3}2^m \right\rceil$ member-sets, it satisfies conjecture 1.*

The proof again relies on the averaging method, although in a more complicated manner than the proof to theorem 5. It can be found in [4].

Be aware that, although condition (3) is *sufficient* for a family to satisfy the conjecture, it is not *necessary.* This immediately indicates the limits of the averaging method. The family $\mathcal{F}$ defined by

$$\mathcal{F} = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$$

is an example for which condition (3) does not hold, although it clearly satisfies the main conjecture.

3.3 Counter-examples

Regardless of whether one tries to prove or disprove the union-closed sets conjecture, the notion of a counter-example provides valuable results.

In particular, if we assume the main conjecture to be false, we can investigate the properties of a *minimal counter-example*, where we minimize over the number of member-sets in a family. In other words: if $\mathcal{F}$ is to be a minimal counter-example with m member-sets, then every family with less than m member-sets should satisfy the conjecture.

In addition, we require that a minimal counter-example be separating. Then we cannot just duplicate elements and obtain an arbitrarily big ground set for essentially the same family. See also the discussion following definition 11 as to why we may always assume separation.

Proposition 9. *Let $\mathcal{F}$ be a minimal counter-example. Then $|\mathcal{F}|$ is odd.*

Proof. Suppose $|\mathcal{F}| = 2n$ for some $n \in \mathbb{N}$ in order to reach a contradiction. Then the maximal frequency in $\mathcal{F}$ is bounded above by $n-1$. Consider the family $\mathcal{F}' \coloneqq \mathcal{F} \setminus \{A\}$ obtained by removing a basis set $A \in \mathcal{B}$ from $\mathcal{F}$. Proposition 2 implies that this family is union-closed. It is also non-trivial, otherwise $\mathcal{F} = \{A\}$ or $\mathcal{F} = \{\emptyset, A\}$, both of which are not counter-examples. We have $|\mathcal{F}'| = 2n-1$ and the maximal frequency in $\mathcal{F}'$ is again bounded above by $n - 1 < \frac{2n-1}{2}$. But then $\mathcal{F}'$ is a strictly smaller counter-example than $\mathcal{F}$, which contradicts the minimality of $\mathcal{F}$. □

We can also determine the frequency of the most occurring element in a minimal counter-example:

Proposition 10. *Let $\mathcal{F}$ be a minimal counter-example with $|\mathcal{F}| = 2n+1$ for some $n \in \mathbb{N}$. Then there exists an element $x \in X$ that exactly has frequency n.*

Proof. Clearly the maximal frequency in $\mathcal{F}$ is less than or equal to n. Again, consider the union-closed family $\mathcal{F}' \coloneqq \mathcal{F} \setminus \{A\}$ obtained by removing a basis set $A \in \mathcal{B}$ from $\mathcal{F}$. Some element $x \in X$ is common in $\mathcal{F}'$ and thus appears in at least n member-sets. As a result, x also appears at least n times in $\mathcal{F}$. □

In other words, a minimal counter-example 'almost' satisfies the main conjecture, in the sense that there exists an element that is only one member-set away from being common. In fact, as is also the subject of a short article by Norton and Servate [5], this property may be strengthened:

Proposition 11. *Let $\mathcal{F}$ be a minimal counter-example with $|\mathcal{F}| = 2n+1$ for some $n \in \mathbb{N}$. Then there exist at least three distinct elements $x, y, z \in X$ that have exactly frequency n.*

Proof. The first element x follows from the previous result. Let $A \in \mathcal{B}$ be a basis set that contains x and consider the union-closed family $\mathcal{F}' \coloneqq \mathcal{F} \setminus \{A\}$. The existence of such a member-set is guaranteed by corollary 1. Then some element y has $|\mathcal{F}'_y| = n$ and therefore $|\mathcal{F}_y| = n$. In particular, $y \notin A$, otherwise we would have $|\mathcal{F}_y| = n + 1$. Hence, x is distinct from y.

Additionally, let $B \in \mathcal{B}$ be a basis set that contains y. This time consider the union-closed family $\mathcal{F}'' := \mathcal{F} \setminus \{A, B\}$. Again, $\mathcal{F}''$ is non-trivial. Then $|\mathcal{F}''| = 2n - 1$, so some element z has $|\mathcal{F}''_z| = n$ and therefore $|\mathcal{F}_z| = n$. From the same argument as before, $z \notin A, B$. Hence, z is distinct from x and y. $\square$

Naturally, we wonder whether we can find a fourth element of maximal frequency. Unfortunately, this seems to be a more difficult question. In some cases, a similar method as above suffices: if we can remove two basis sets so that every known element of maximal frequency is contained at least once, the existence of another such element is implied. However, the structure of other families turns out to be more complex.

Hu [6] proves that a minimal counter-example $\mathcal{F}$ on m elements contains at least $4m-1$ member-sets using propositions 9 and 10. Let us say $X = \{1, ..., m\}$ and assume without loss of generality that $|\mathcal{F}_1| \leq \cdots \leq |\mathcal{F}_m|$. For all elements $i \in X$, define the set

$$A_i := \bigcup_{A \in \mathcal{F}_{\bar{i}}} A.$$

Note that $i \notin A_i$. Additionally, $j \in A_i$ for all $j > i$. Indeed, observe that $\mathcal{F}_j$ is the disjoint union of $\mathcal{F}_{ij}$ and $\mathcal{F}_{\bar{i}j}$. If $\mathcal{F}_{\bar{i}j}$ were empty, we would have

$$\mathcal{F}_j = \mathcal{F}_{ij} \subset \mathcal{F}_i$$

with strict inequality by the separating property of $\mathcal{F}$. But then we would have $|\mathcal{F}_j| < |\mathcal{F}_i|$, a contradiction. Thus $\mathcal{F}_{\bar{i}j}$ is non-empty and therefore $j \in A_i$.

By construction, every A_i is a member-set of $\mathcal{F}$. Hence $\mathcal{F}$ contains a sub-family

$$\mathcal{G} := \{X, A_1, ..., A_{m-1}\}$$

where m is contained in every member-set of $\mathcal{G}$, i.e. $\mathcal{G} \subseteq \mathcal{F}_m$. Consequently, every element's frequency in $\mathcal{F}$ is bounded below by its frequency in $\mathcal{F}_{\overline{m}}$ plus its frequency in $\mathcal{G}$. Let us say an element i *dominates* an element j in $\mathcal{F}$ if it appears in every member-set of $\mathcal{F}_j$. We then have the following results for $\mathcal{F}$:

Lemma 1. [6] *Let $i \in \{1, ..., m-1\}$ such that $|\mathcal{G}_i| < m - 1$. Then there exists an element $j \in \{1, ..., m-1\}$ that dominates i in $\mathcal{F}$ such that $|\mathcal{G}_j| = m - 1$.*

Proof. If i has a frequency less than $m - 1$ in $\mathcal{G}$ then there exists an element $j \in \{i+1, \ldots, m-1\}$ with $i \notin A_j$. Hence j dominates i in $\mathcal{F}$ and $|\mathcal{F}_j| > |\mathcal{F}_i|$, by the separating property of $\mathcal{F}$. If $|\mathcal{G}_j| = m - 1$ we are done. Otherwise we may repeat the above process until we find an element k that dominates i in $\mathcal{F}$ such that $|\mathcal{G}_k| = m - 1$. $\square$

Corollary 2. [6] *Let $\mathcal{F}'$ be a non-trivial sub-family of $\mathcal{F}_{\overline{m}}$. Among the elements of maximal frequency in $\mathcal{F}'$, one has frequency $m - 1$ in $\mathcal{G}$.*

Proof. Let i be an element of maximal frequency in $\mathcal{F}'$. If $|\mathcal{G}_i| < m-1$, we find an element j that dominates i in $\mathcal{F}$ and therefore in $\mathcal{F}'$ such that $|\mathcal{G}_j| = m-1$ by lemma 1. □

Theorem 7. [6] *Let $\mathcal{F}$ be a minimal counter-example on m elements. Then $|\mathcal{F}| \geq 4m-1$.*

Proof. From proposition 9 we have $|\mathcal{F}| = 2n+1$ for some $n \in \mathbb{N}$. Then proposition 10 implies $|\mathcal{F}_m| = n$ and hence $|\mathcal{F}_{\overline{m}}| = n+1$. Hence there exists an element i with at least frequency $\frac{n+1}{2}$ in $\mathcal{F}_{\overline{m}}$ and frequency $m-1$ in $\mathcal{G}$ by corollary 2. Then we have

$$\frac{n+1}{2} + m - 1 \leq |\mathcal{F}_i| \leq n$$

where the right inequality holds because $\mathcal{F}$ is a counter-example. As a result

$$|\mathcal{F}| = 2n+1 \geq 4m-1,$$

which finishes the proof. □

Živković and Vučković [7] present a computer assisted proof that a minimal counter-example must contain at least 13 elements. Hence we find:

Corollary 3. *The main conjecture holds for all union-closed families of at most 50 member-sets.*

4 Equivalent formulations

The union-closed sets conjecture knows several equivalent formulations in various fields of mathematics. Although some of these forms have proven more fruitful than others, all of them offer new perspectives and their own approach to the conjecture. In this chapter, we cover some of the equivalent conjectures and interesting results that they have provided.

4.1 Intersection-closed families

Naturally, the idea of union-closed families inspires that of intersection-closed families.

Definition 14. A family $\mathcal{F}$ is *intersection-closed* if for any $A, B \in \mathcal{F}$ we have $A \cap B \in \mathcal{F}$.

A straightforward equivalent conjecture then follows, which is stated by Bruhn and Schaudt [1] as follows:

Conjecture 2. *Any finite intersection-closed family $\mathcal{F}$ of more than one member-set contains a rare element.*

In order to prove that conjecture 2 is equivalent to the main conjecture, we first introduce the notion of a dual family:

Definition 15. The *dual family* of a family $\mathcal{F}$ on a ground set X is defined by

$$\hat{\mathcal{F}} \coloneqq \{X \setminus A : A \in \mathcal{F}\}.$$

Figure 3 shows an example of an intersection-closed family. In particular, it is the dual family of the union-closed family in figure 1.

1234
124 123
14 13 12
4 3 1
∅

Figure 3: An example of an intersection-closed family.

Let a family $\mathcal{F}$ and its dual $\hat{\mathcal{F}}$ be given. If

$$X(\mathcal{F}) = X(\hat{\mathcal{F}}), \tag{4}$$

then the dual family of $\hat{\mathcal{F}}$ is again $\mathcal{F}$ itself. This equation holds unless some element $x \in X$ appears in every member-set of $\mathcal{F}$, in which case it does not appear in any of the member-sets of $\hat{\mathcal{F}}$. In this light, let us call a family is *simple* if condition (4) holds.

Proposition 12. *Let $\mathcal{F}$ be a simple family and $x \in X$. Then x is common in $\mathcal{F}$ if and only if x is rare in $\hat{\mathcal{F}}$.*

Proof. It is a direct consequence from definition 15 that $|\mathcal{F}| = |\hat{\mathcal{F}}|$ as well as $|\mathcal{F}_x| = |(\hat{\mathcal{F}})_{\overline{x}}|$. The result follows. □

Proposition 13. *Let $\mathcal{F}$ be a family.*

(i) *If $\mathcal{F}$ is union-closed, then $\hat{\mathcal{F}}$ is intersection-closed.*

(ii) *If $\mathcal{F}$ is intersection-closed, then $\hat{\mathcal{F}}$ is union-closed.*

Proof. For (i), let $\mathcal{F}$ be union-closed and $A, B \in \hat{\mathcal{F}}$. Then both $X \setminus A$ and $X \setminus B$ are member-sets of $\mathcal{F}$. So $X \setminus (A \cap B) \in \mathcal{F}$ by union-closedness of $\mathcal{F}$. As a result, $A \cap B \in \hat{\mathcal{F}}$, which implies intersection-closedness of $\hat{\mathcal{F}}$. The proof for (ii) is analogous. □

Theorem 8. *Conjecture 2 is equivalent to the main conjecture.*

Proof. Let $\mathcal{F}$ be a non-trivial union-closed family. Proposition 13 implies that its dual family $\hat{\mathcal{F}}$ is intersection-closed. From proposition 5 we may assume without loss of generality that $\emptyset \in \mathcal{F}$, which implies $\mathcal{F}$ is simple. Then, $\mathcal{F}$ contains more than one member-set and therefore so does $\hat{\mathcal{F}}$. Now, assuming the intersection-closed sets conjecture is true, there exists $x \in X$ that is rare in $\hat{\mathcal{F}}$. Proposition 12 then implies that x is common in $\mathcal{F}$.

Conversely, let $\mathcal{F}$ be an intersection-closed family with more than one member. Proposition 13 implies that its dual family $\hat{\mathcal{F}}$ is union-closed. Since $\mathcal{F}$ contains more than one member-set, $\hat{\mathcal{F}}$ is non-trivial. Now, assuming the union-closed sets conjecture is true, there exists $x \in X$ that is common in $\hat{\mathcal{F}}$. Finally, we may assume without loss of generality that $\mathcal{F}$ is simple: if some element x appears in every member-set of $\mathcal{F}$, then $\mathcal{F}$ satisfies conjecture 2 if $\mathcal{F} - x$ does. Proposition 12 then implies that x is rare in $\mathcal{F}$. □

Note that it follows from the proof of theorem 8 that we may always assume union-closed and intersection-closed families to be simple without loss of generality w.r.t their respective conjectures.

Many of the results for union-closed families can be easily translated to an intersection-closed form. For example, consider the intersection-closed form of proposition 5:

Proposition 14. *Let $\mathcal{F}$ be an intersection-closed family of more than one member-set and $X \notin \mathcal{F}$. Then $\mathcal{F}$ satisfies conjecture 2 if $\mathcal{F} \cup \{X\}$ does.*

Proof. Let $x \in X(\mathcal{F})$. Then the frequency of x in $\mathcal{F} \cup \{X\}$ is equal to the frequency of x in $\mathcal{F}$ plus one. Moreover, the family $\mathcal{F} \cup \{X\}$ contains one more member-set than $\mathcal{F}$. Then x is rare in $\mathcal{F}$ if x is rare in $\mathcal{F} \cup \{X\}$. □

Finally, we take a look at the complement of a union-closed family. Given a set S, denote by 2^S the power set of S. For any family $\mathcal{F}$ on a ground set X we then have $\mathcal{F} \subseteq 2^X$, so that we may consider the complement of $\mathcal{F}$ in 2^X.

Definition 16. The *complement family* of a family $\mathcal{F}$ with ground set X is defined by

$$\mathcal{F}^c := 2^X \setminus \mathcal{F}.$$

The complement of a union-closed family is generally not union-closed (or intersection-closed). However, we obtain a similar result to proposition 12.

Proposition 15. *Let $\mathcal{F}$ be a simple family and $x \in X$. Then x is common in $\mathcal{F}$ if and only if x is rare in $\mathcal{F}^c$.*

Proof. Every element appears in exactly half of the member-sets of 2^X, which is the disjoint union of $\mathcal{F}$ and $\mathcal{F}^c$. Therefore we get

$$|\mathcal{F}_x| + |\mathcal{F}^c_x| = \frac{1}{2}|2^X| = |\mathcal{F}_{\overline{x}}| + |\mathcal{F}^c_{\overline{x}}|$$

and thus $|\mathcal{F}_x| \geq |\mathcal{F}_{\overline{x}}|$ if and only if $|\mathcal{F}^c_x| \leq |\mathcal{F}^c_{\overline{x}}|$. □

4.2 Lattice theory

Recall that we assume all families to be finite. Then one approach to the union-closed sets conjecture is to consider families as finite lattices under inclusion.

Definition 17. A *finite lattice* is a finite poset $(L, \leq)$ in which every pair $a, b \in L$ of elements has a unique greatest lower bound $a \wedge b$ (the *meet*) and a unique least upper bound $a \vee b$ (the *join*).

Observe that a finite lattice always has a unique minimal and a unique maximal element, which we may refer to as *zero* and *one*, respectively.

Let $\mathcal{F}$ be a union-closed family. From proposition 5, we may assume the empty set to be a member-set of $\mathcal{F}$. Bruhn and Schaudt [1] point out that $(\mathcal{F}, \subseteq)$ then forms a lattice. The unique least upper bound for any $A, B \in \mathcal{F}$ is $A \vee B = A \cup B \in \mathcal{F}$, while $\emptyset \in \mathcal{F}$ guarantees that A and B always have a greatest lower bound. Such a greatest lower bound is unique: suppose $R, S \in \mathcal{F}$ are two lower bounds, then $R \cup S \in \mathcal{F}$ is also a lower bound. Analogously, an intersection-closed family that contains the ground set as a member-set also forms a lattice under inclusion.

Figure 4 shows the lattice under inclusion of the example in figure 1.

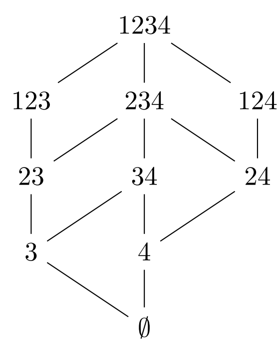

Figure 4: The lattice of the union-closed family of figure 1.

Definition 18. Let L be a lattice. A non-zero element $a \in L$ is *join-irreducible* if there are no elements $b, c \in L \setminus \{a\}$ such that $b \vee c = a$.

In a union-closed family $\mathcal{F}$, the join-irreducible elements are exactly the basis sets (see definition 7). We define a *meet-irreducible* element analogously.

In addition, given a lattice L and $a \in L$ we define $[a) \coloneqq \{x \in L : x \geq a\}$. Bruhn and Schaudt [1] prove that the following conjecture is equivalent to the main conjecture:

Conjecture 3. *Let L be a finite lattice with at least two elements. Then there is a join-irreducible element a with $|[a)| \leq \frac{1}{2}|L|$.*

Theorem 9. [1] *Conjecture 3 is equivalent to the main conjecture.*

We shall prove that conjecture 3 is equivalent to conjecture 2. The result then follows from theorem 8.

Proof. Let $\mathcal{F}$ be an intersection-closed family of more than one member-set and consider the lattice $(\mathcal{F}, \subseteq)$. From proposition 14, we may assume the ground set X to be a member-set of $\mathcal{F}$. Now, assuming conjecture 3 holds, there exists a join-irreducible member-set $J \in \mathcal{F}$ such that $|[J)| \leq \frac{1}{2}|\mathcal{F}|$.

We show that there exists an element $x \in J$ that does not lie in any proper subset of J (in $\mathcal{F}$): if such an element does not exist, then J is the union of all its proper subsets. But then J is exactly the least upper bound over all those proper subsets, which contradicts that J is join-irreducible. Hence there is indeed an $x \in J$ that is not contained in any proper subset of J.

Next, consider $A \in \mathcal{F}$ that contains x. Then $J \cap A$ is a member-set of $\mathcal{F}$ by intersection-closedness. Because $J \cap A$ is a subset of J containing x, it must be equal to J. Therefore $J \subseteq A$, which implies $A \in [J)$. Since $|[J)| \leq \frac{1}{2}|\mathcal{F}|$, it follows that x appears in at most half of the member-sets of $\mathcal{F}$.

Conversely, consider a lattice $(L, \leq)$ with at least two elements. Associate with every $x \in L$ the set $J(x)$ of join-irreducible elements $z \leq x$. Then for $x, y \in L$ we have that $J(x) \cap J(y) = J(x \wedge y)$ and so the family

$$\mathcal{F} \coloneqq \{J(x) : x \in L\}$$

is intersection-closed.

Let $x, y \in L$. Then $x \leq y$ implies $J(x) \subseteq J(y)$. Conversely, $J(x) \nsubseteq J(y)$ implies some join-irreducible $z \leq x$ exists such that $z \nleq y$. Therefore we cannot have $x \leq y$, as transitivity would then imply $z \leq y$. Hence, $J(x) \subseteq J(y)$ if and only if $x = y$. By symmetry we have $J(x) = J(y)$ if and only if $x = y$. As a result we have $|\mathcal{F}| = |L|$. The ground set of $\mathcal{F}$ consists of exactly the join-irreducible elements in L.

Now, assuming conjecture 2 holds, we obtain a join-irreducible $x \in L$ that appears in at most half the member-sets of $\mathcal{F}$. Finally, for $y \in L$ we have $y \geq x$ if and only if $x \in J(y)$, which implies $|[x)| = |\mathcal{F}_x| \leq \frac{1}{2}|L|$.

As a result, conjecture 3 is equivalent to conjecture 2 and therefore to the main conjecture. □

The lattice theoretic form of the main conjecture has been verified for a number of types of lattices. Bruhn and Schaudt [1] say that the strongest result is due to Reinhold [8] and provide a modified version of the proof, which we cover below.

Let L be a lattice and $x, y \in L$ such that $x < y$. Then x is a *lower cover* for y if $x \leq z \leq y$ implies $x = z$ or $y = z$ for all elements z. Intuitively, there is no element 'between' x and y. A lattice L is *lower semimodular* if $a \wedge b$ is a lower cover of $a \in L$, whenever $b \in L$ is a lower cover of $a \vee b$. For an example, see figure 5.

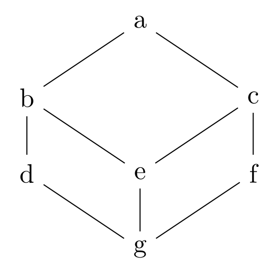

Figure 5: An example of a lower semimodular lattice.

Theorem 10. [1] *Conjecture 3 holds for lower semimodular lattices.*

Proof. Let L be a lower semimodular with at least two elements. If the maximal element $1 \in L$ is join-irreducible, conjecture 3 is trivially satisfied. If not, consider $a, b \in L$ such that b is a lower cover of 1 and a is join-irreducible with $a \not\leq b$. Note that such an a is guaranteed to exist, otherwise b is an upper bound over all elements in $L \setminus \{1\}$ and then 1 would be join-irreducible. Note also that $1 = a \vee b$. Now consider the following mapping:

$$\varphi : [a) \longrightarrow L \setminus [a)$$
$$x \longmapsto x \wedge b$$

We will show that φ is an injection, from which $|[a)| \leq \frac{1}{2}|L|$ directly follows. First, observe that $a \not\leq b$ implies $a \not\leq x \wedge b$, so that indeed $x \wedge b \in L \setminus [a)$. To see that φ is injective, suppose there exist distinct $x, y \in [a)$ such that $x \wedge b = y \wedge b$. We have $x \wedge y < x$ or $x \wedge y < y$ (or both). Let us assume the former by symmetry. This implies

$$x \wedge b = x \wedge y \wedge b \leq x \wedge y < x.$$

Because L is semimodular and b is a lower cover of $1 = x \vee b$, we obtain that $x \wedge b$ is a lower cover of x. With the above inequalities this implies $x \wedge b = x \wedge y$. Therefore,

$$a \leq x \wedge y = x \wedge b \leq b.$$

However, this contradicts $a \not\leq b$ and we find that φ is injective. □

4.3 Graph theory

An equivalent form in the field of graph theory is due to Bruhn et al. [9]. In this section we discuss the formulation and proof of its equivalence as presented in that article. All graphs are assumed to be simple, i.e. they contain no loops or multiple edges.

Definition 19. A *bipartite graph* $G = (V, E)$ is a graph whose vertex set V can be partitioned into two sets U and W, such that every edge $e \in E$ connects a vertex in U to a vertex in W.

We refer to U and W as the *bipartition classes* of G. Note that if G is connected, this partition is unique.

Definition 20. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is *stable* (or *independent*) if no two vertices in S are adjacent. A stable set is *maximal* if it is maximal under inclusion.

In other words, a stable set S is maximal if every vertex outside S has a neighbour in S.

Definition 21. Let $G = (V, E)$ be a graph. Then we define the family of all maximal stable sets by

$$\mathcal{A}(G) \coloneqq \{S \subseteq V : S \text{ is a maximal stable set}\}.$$

If no confusion can arise, we may also simply refer to $\mathcal{A}(G)$ as $\mathcal{A}$. Recall that for a vertex v, we denote by $\mathcal{A}_v$ the sub-family of $\mathcal{A}$ with all member-sets that contain v (conform to definition 4). We can then extend the definition of common and rare elements to vertices in a graph:

Definition 22. A vertex v is *common* in a graph G if $|\mathcal{A}_v| \geq \frac{1}{2}|\mathcal{A}|$.

Definition 23. A vertex v is *rare* in a graph G if $|\mathcal{A}_v| \leq \frac{1}{2}|\mathcal{A}|$.

We can now state the main conjecture in the language of graph theory:

Conjecture 4. *Let G be a finite bipartite graph with at least one edge. Then each of its two bipartition classes contains a vertex that is rare in G.*

But why do we specifically require graphs to be bipartite here? Because all other graphs would trivially satisfy the conjecture. Indeed, a well-known result in graph theory states that a graph is bipartite if and only if it contains no cycles of odd length. Moreover, any two adjacent vertices cannot both be contained in the same stable set, so that at least one of them must be rare. Then on an odd cycle in a non-bipartite graph, we obtain two adjacent vertices that are both rare, and the conjecture is satisfied.

In order to prove that conjecture 4 is equivalent to the main conjecture, we first require the notion of a neighbourhood and some auxiliary statements:

Definition 24. Given a graph $G = (V, E)$ and a set $S \subseteq V$ of vertices, we define the *neighbourhood* $N(S)$ as the set of all vertices adjacent to some $v \in S$.

Proposition 16. *Let G be a graph with bipartition classes U, W and let S be a maximal stable set. Then*

$$U \setminus S = N(W \cap S).$$

Proof. Let $v \in U \setminus S$. Then $v \notin S$, which implies that v is adjacent to some $u \in S$ as S is a maximal stable set. Since G is a bipartite graph we must then have $u \in W$. Then $u \in W \cap S$ and thus $v \in N(W \cap S)$.

Conversely, let $v \in N(W \cap S)$. Then there exists a vertex $u \in W \cap S$ that is adjacent to v. Since G is a bipartite graph, we must have $v \in U$. Moreover, $u \in S$ implies $v \notin S$ since S is a stable set. So $v \in U \setminus S$ and the equality follows. □

Using this identity we can easily prove the following statement; it says that a maximal stable set S is determined by $U \cap S$ (or alternatively by $W \cap S$).

Proposition 17. [9] *Let G be a graph with bipartition classes U, W and let S be a maximal stable set. Then*

$$S = (U \cap S) \cup (W \setminus N(U \cap S)).$$

Proof. Since U and W partition G, it holds that $S = (U \cap S) \cup (W \cap S)$. Proposition 16 gives $W \setminus S = N(U \cap S)$. Hence

$$W \setminus N(U \cap S) = W \setminus (W \setminus S) = W \cap S$$

and the result follows. □

Proposition 18. [9] *Let G be a graph with bipartition classes U, W and let S, T be maximal stable sets. Then*

$$R \coloneqq (U \cap S \cap T) \cup (W \setminus N(U \cap S \cap T))$$

is a maximal stable set.

Proof. None of the vertices in $U \cap S \cap T \subseteq U$ are adjacent to any of the vertices in $W \setminus N(U \cap S \cap T) \subseteq W$. Because U and W are the bipartition classes of G, this means no two vertices in R are adjacent to each other. Hence R is a stable set.

Moreover, R is maximal: let $v \notin R$. We distinguish two cases, either $v \in U \setminus R$ or $v \in W \setminus R$. In the latter case, $v \in N(U \cap S \cap T)$ and therefore $v \in N(R)$. In the former case, $v \notin S$ or $v \notin T$. By symmetry, let us assume $v \notin S$. Since S is maximal, v has a neighbour $u \in S$. In particular, $u \in W$ because G is bipartite. But $u \notin N(S \cap T)$ because S is stable. As a result $u \in W \setminus N(S \cap T)$ and so $v \in N(R)$. Consequently, R is indeed a maximal stable set. □

We are now able to prove that the graph theoretic formulation is equivalent to the original statement of the main conjecture.

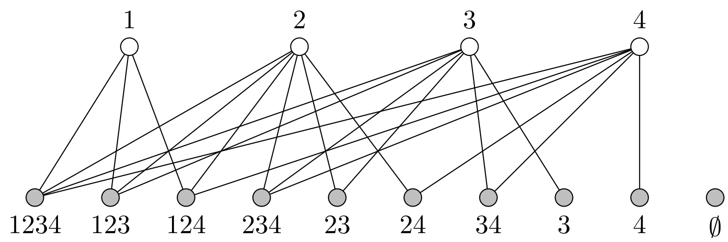

Figure 6: The bipartite graph representing the union-closed family of figure 1.

Theorem 11. [9] *Conjecture 4 is equivalent to the main conjecture.*

Proof. Let $\mathcal{F}$ be a non-trivial union-closed family. Define a bipartite graph G with bipartition classes $\mathcal{F}$ and X, where $x \in X$ is adjacent to $A \in \mathcal{F}$ if and only if $x \in A$. The former class then contains vertices that correspond to the member-sets of $\mathcal{F}$ and the latter contains vertices that correspond to the elements that those member-sets consist of. We refer to the vertices simply by the name of the member-sets/elements they represent. See figure 6 for an illustration.

Consider the mapping of the family $\mathcal{A}$ of maximal stable sets onto the union-closed family $\mathcal{F}$ defined by

$$\begin{aligned} \tau : \mathcal{A} &\longrightarrow \mathcal{F} \\ S &\longmapsto X \setminus S. \end{aligned}$$

We will prove that τ is a bijection. First we show that indeed $\tau(S) \in \mathcal{F}$ for all $S \in \mathcal{A}$, so that τ is well-defined. From proposition 16 we have

$$X \setminus S = N(\mathcal{F} \cap S).$$

Moreover, from the construction of G it is seen that $N(\mathcal{F} \cap S)$ is just the vertex set of all elements that appear in some member-set of $\mathcal{F} \cap S$, i.e.

$$N(\mathcal{F} \cap S) = \bigcup_{A \in \mathcal{F} \cap S} A,$$

which itself is a member-set of $\mathcal{F}$ by the union-closed property of $\mathcal{F}$. By proposition 5, we may assume that $\emptyset$ is a member-set of $\mathcal{F}$. As a result, $\tau(S) = X \setminus S \in \mathcal{F}$.

To see that τ is injective, let $S, T \in \mathcal{A}$. Then proposition 17 implies that $X \setminus S = X \setminus T$ if and only if $S = T$. For surjectivity, consider a member-set $A \in \mathcal{F}$ and denote by $A' \coloneqq X \setminus A$ the vertex set of elements that don't appear in A. The vertex set S defined by

$$S \coloneqq A' \cup (\mathcal{F} \setminus N(A'))$$

is a stable set. Moreover, S is maximal: let $v \notin S$. Then either $v \in A \subseteq X$ or $v \in N(A') \subseteq \mathcal{F}$. In the latter case, $A' \subseteq S$ implies $v \in N(S)$. In the former case, v is adjacent to $A \in \mathcal{F}$ by construction of G. Also, $A \in \mathcal{F}$ is not adjacent to any vertex in $A' \subseteq X$. As a result, $A \in \mathcal{F} \setminus N(A') \subseteq S$ and $v \in N(S)$. So S is indeed a maximal stable set, i.e. $S \in \mathcal{A}$. Finally, it is easily seen that

$$\tau(S) = X \setminus S = X \setminus A' = A.$$

From injectivity and surjectivity we have that τ is a bijection, and therefore $|\mathcal{A}| = |\mathcal{F}|$. Observe also that $x \in S$ if and only if $x \notin \tau(S)$ for every $x \in X$. Now, assuming conjecture 4 is true, there exists a vertex $v \in X$ which is rare in G. As a result,

$$|\mathcal{F}_v| = |\mathcal{A}_{\overline{v}}| \geq \frac{1}{2}|\mathcal{A}| = \frac{1}{2}|\mathcal{F}|$$

and v is common in $\mathcal{F}$. The main conjecture then follows.

Conversely, let G be a bipartite graph with bipartition classes U, W and at least one edge. By symmetry it is sufficient to find a vertex in U that is rare in G. Define the family $\mathcal{F}$ by

$$\mathcal{F} \coloneqq \{U \setminus S : S \in \mathcal{A}\}.$$

There is a non-isolated vertex $v \in U$, which implies there exists a maximal stable set that does not contain v. As a result, $\mathcal{F}$ contains at least one non-empty member-set and is non-trivial.

Moreover, $\mathcal{F}$ is union-closed: let $A, B \in \mathcal{F}$. Then there exist $S, T \in \mathcal{A}$ such that $U \setminus S = A$ and $U \setminus T = B$. From proposition 18, we know that there exists a maximal stable set $R \in \mathcal{A}$ such that $U \setminus (S \cap T) = U \setminus R \in \mathcal{F}$. So $A \cup B \in \mathcal{F}$ and $\mathcal{F}$ is indeed union-closed.

Define again the bijection τ as before, from $\mathcal{A}$ to $\mathcal{F}$ with the property that $x \in S$ if and only if $x \notin \tau(S)$ for every $x \in X$. Now, assuming the main conjecture true, there exists an element $x \in X$ which is common in $\mathcal{F}$. From the same argument as before this implies x is rare in G. Conjecture 4 then follows. □

Bruhn et al. prove that several types of bipartite graphs satisfy the graph theoretic form of the main conjecture. For this, they often rely on a local property. As an intermediate result, let us first consider the following:

Lemma 2. [9] *Let x be a vertex in a bipartite graph G. Then $|\mathcal{A}_{N(x)}| \leq |\mathcal{A}|$.*

Proof. We construct an injection from $\mathcal{A}_{N(x)}$ into $\mathcal{A}_x$. Consider the mapping defined by

$$\begin{aligned}\tau : \mathcal{A}_{N(x)} &\longrightarrow \mathcal{A}_x \\ S &\longmapsto (S \setminus L_1) \cup \{x\} \cup (L_2 \setminus N(S \cap L_3)),\end{aligned}$$

where L_i denotes the set of vertices at distance i to x. First, let us see that we indeed have $\tau(S) \in \mathcal{A}_x$. It is clear that x is not adjacent to any vertex in $\tau(S)$ because no vertex of L_1 is contained in $\tau(S)$. Moreover, $N(L_2) = L_1 \cup L_3$ and in particular

$$N(L_2 \setminus N(S \cap L_3)) = L_1 \cup (L_3 \setminus S),$$

of which no element is contained in $\tau(S)$. Thus, no two vertices in $\tau(S)$ are adjacent to each other and $\tau(S)$ is stable. In addition, $\tau(S)$ is maximal: let v be any vertex that is not contained in $\tau(S)$. If $v \in L_1$ then $v \in N(x)$. If $v \in L_2$ then $v \in N(S \cap L_3)$. If $v \in L_i$ for some $i \geq 3$, note that v is not contained in S. Because S is a maximal stable set, this means v has a neighbour $u \in S$. In particular, $u \notin L_1$ so that $u \in \tau(S)$. Therefore $\tau(S) \in \mathcal{A}_x$.

For injectivity, suppose $\tau(S) = \tau(T)$ for $S, T \in \mathcal{A}_{N(x)}$. Then S and T are identical outside $L_1 \cup L_2$. However, S and T are also identical on $L_1 \cup L_2$: from $L_1 = N(x)$ we know that L_1 lies in both S and T. Second, since $L_2 \subseteq N(L_1)$, no element of L_2 can be contained in S or T. Thus $S = T$ and τ is an injection. As a result, $|\mathcal{A}_{N(x)}| \leq |A_x|$. □

From this, we obtain a lemma that can be applied to show that a graph satisfies conjecture 4, even though only a local requirement needs to be satisfied:

Lemma 3. [9] *Let x, y be two adjacent vertices in a bipartite graph $\mathcal{G}$ with $N^2(x) \subseteq N(y)$. Then y is rare.*

Proof. Suppose $N^2(x) \subseteq N(y)$ and let $S \in \mathcal{A}_y$ be any maximal stable set that contains y. Then none of the vertices in $N(y)$ appear in S. Every vertex in $N(x)$ thus has no neighbour in S and therefore must be contained in S. Hence $N(x) \subseteq S$, i.e. $S \in \mathcal{A}_{N(x)}$. Since S was arbitrary, we therefore obtain $\mathcal{A}_y \subseteq \mathcal{A}_{N(x)}$ and so $|\mathcal{A}_y| \leq |\mathcal{A}_{N(x)}|$. Together with lemma 2, this implies $|\mathcal{A}_y| \leq |\mathcal{A}_x|$.

Because x and y are adjacent, every stable set contains at most one of the two. Thus

$$|\mathcal{A}_x| + |\mathcal{A}_y| \leq |\mathcal{A}|.$$

It follows that $|\mathcal{A}_y| \leq \frac{1}{2}|\mathcal{A}|$, which finishes the proof. □

In particular, consider a special case: let $\mathcal{F}$ be a family that contains a singleton $\{x\}$ and let G be the graph obtained by the process described in the first part of the proof of theorem 11. By construction, the only vertex adjacent to $\{x\}$ is x. Thus $N^2(\{x\}) \subseteq N(x)$, which implies that the vertex x is rare in G by lemma 3. From the same reasoning as in the proof of theorem 11, the element x must then be common in $\mathcal{F}$. Lemma 3 therefore implies theorem 3.

In fact, we may generalize lemma 3 to two adjacent vertices:

Lemma 4. [9] *Let G be a bipartite graph. Let y and z be two neighbours of a vertex x such that $N^2(x) \subseteq N(y) \cup N(z)$. Then at least one of y and z is rare.*

Proof. By symmetry we may assume $|\mathcal{A}_{y\overline{z}}| \leq |\mathcal{A}_{\overline{y}z}|$. From a similar argument to that in the proof of lemma 3, we have that $N^2(x) \subseteq N(y) \cup N(z)$ implies $|\mathcal{A}_{yz}| \leq |\mathcal{A}_{N(x)}|$. Note that $\mathcal{A}_x \subseteq \mathcal{A}_{\overline{yz}}$ so that $|\mathcal{A}_x| \leq |\mathcal{A}_{\overline{yz}}|$. Again together with lemma 2, this implies $|\mathcal{A}_{yz}| \leq |\mathcal{A}_{\overline{yz}}|$. Then

$$|\mathcal{A}_y| = |\mathcal{A}_{y\overline{z}}| + |\mathcal{A}_{yz}| \leq |\mathcal{A}_{\overline{y}z}| + |\mathcal{A}_{\overline{yz}}| = |\mathcal{A}_{\overline{y}}|$$

and thus y is rare. □

In turn, this corresponds to theorem 4. For similar reasons as in section 3.1 we cannot generalize this result any further.

Let us now cover four results for sub-classes of bipartite graphs. First, a bipartite graph is *chordal bipartite* if every cycle of length at least six contains a *chord*, i.e. an edge that connects two vertices that are non-consecutive in the cycle.

Theorem 12. [9] *Conjecture 4 holds for chordal bipartite graphs.*

Proof. A vertex v in a bipartite graph is called *weakly simplicial* if the neighbourhoods of its neighbours form a chain under inclusion. In other words, if $N(v)$ contains n vertices, then we can label those in such a way that

$$N(v_1) \subseteq N(v_2) \subseteq \cdots \subseteq N(v_n).$$

But then $N^2(v) = N(v_n)$, and v_n is a rare vertex in G by lemma 3. Pelsmajer, Tokaz and West [10] prove that a weakly simplicial vertex is always contained in each of the bipartition classes of a chordal bipartite graph with at least one edge. Hence chordal bipartite graphs satisfy conjecture 4. □

Second, a *circular interval graph* is defined as follows: let a finite subset of points on a circle be the vertex set. Then for a given set of intervals on the circle, consider two vertices to be adjacent if there is an interval containing them both.

We obtain a bipartite graph if we partition the vertices into two bipartition classes and delete every edge with both its end vertices in the same class. Such a graph is called a *bipartitioned circular interval graph*.

Theorem 13. [9] *Conjecture 4 holds for bipartitioned circular interval graphs.*

Proof. Consider a bipartitioned circular interval graph with interval set $\mathcal{I}$ and let v be a non-isolated vertex.

For every neighbour $u \in N(v)$, choose some interval $I_u \in \mathcal{I}$ that contains both u and v. If the union of these intervals covers the entire circle, consider the two intervals I_y and I_z that reach the furthest from v in anti-clockwise and clockwise order respectively. Then the circle is covered by $I_y \cup I_z$ alone. Every vertex of $N^2(v)$ falls in the bipartition class that does not contain y and z. Since it also falls in $I_y \cup I_z$, it must thus be adjacent to at least one of y and z. As a result, $N^2(v) \subseteq N(y) \cup N(z)$.

So, assume there exists a point p on the circle that is not covered by any I_u with $u \in N(v)$. Let y be the first neighbour of v next to p in clockwise order, and let z be the first neighbour of v next to p in anti-clockwise order. Then y, u, z appear in clockwise order for every $u \in N(v)$. Moreover, $u' \in I_y \cup I_z$ for every u' such that y, u', z appear in clockwise order.

Now let $u \in N^2(v)$ and let w be a neighbour of both v and u. There must exist an interval $J \in \mathcal{I}$ containing both u and w. If y, u, z appear in clockwise order, we have $u \in I_y \cup I_z$ and hence $u \in N(y) \cup N(z)$. Otherwise, from $w \in N(v)$ we know that y, w, z appear in clockwise order. Thus either y or

z falls in J, from which we obtain that u is adjacent to either y or z. Thus $N^2(v) \subseteq N(y) \cup N(z)$.

In either case, lemma 4 implies that at least one of y and z is rare. Since x is arbitrary, we find a rare vertex in each bipartition class. Henc bipartite circular interval graphs satisfy conjecture 4. □

Third, graph G is *series-parallel* if it does not contain K_4 as a *minor*, i.e. K_4 cannot be obtained from deleting edges and vertices and contracting edges in G.

Theorem 14. [9] *Conjecture 4 holds for bipartite series-parallel graphs.*

Finally, a *subcubic graph* is a graph in which no vertex has a degree greater than 3.

Theorem 15. [9] *Conjecture 4 holds for subcubic graphs.*

We conclude this section with the main result of the paper by Bruhn et al. It illustrates that the graph-theoretic form of the main conjecture also yields results that are relevant to the original statement. In particular, the proof relies on theorem 15:

Theorem 16. [9] *Let $\mathcal{F} \neq \{\emptyset\}$ be a finite union-closed family of sets. If $\mathcal{F}$ is the union-closure of sets of at most size three then it satisfies the main conjecture.*

5 Conclusion

Evidently, the union-closed sets conjecture is far more complicated than it might seem at first glance. Despite the simplicity of elementary set operations, the structure of union-closed families turns out to be deceivingly intricate. As a final remark, to illustrate this complexity, consider the following fun combinatorial exercise my friend Gijs Bellaard proposed:

> *In how many ways can a set S of n elements be expressed as the union of two sets?*

The solution: we count the number of pairs (A, B) such that $A \cup B = S$. In every such pair, each of the n elements has three options; it is contained either in A, in B or in both. Thus we obtain 3^n possible ordered pairs. To find the number of unordered pairs, we note that all pairs except (S, S) have been counted twice. Hence we arrive at a total of $\frac{1}{2}(3^n + 1)$ distinct ways.

Indeed, repeatedly taking the unions over a number of sets quickly yields families that are tough to unravel. Altering a member-set can break the union-closedness or have rippling effects throughout the entire family. It seems that these obstacles are at the heart of this difficult conjecture.

Nonetheless, there are already many remarkable results. Theorems 3, 4, 6, 16 and corollary 3 verify the conjecture for sub-classes of union-closed families and theorems 10, 12, 13, 14, and 15 do the same for sub-classes of the lattice and graph theoretic counterparts of the conjecture. My paper contains a summary of these results on the conjecture and its related concepts. Some observations are my own, such as propositions 3, 8, 15. In addition, I aimed to generally improve the accessibility of proofs and present them in a most intuitive manner to the reader. In some cases I defined new concepts to achieve this. For instance, dual families (and in particular those of simple families) provide a clearer view of the relations between union-closed and intersection-closed families.

Above all, I hope that my paper has inspired the reader to try their own hand at this delightful problem. Perhaps they might even discover its solution...

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